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LETTER TO THE EDITOR

The Wahlquist–Estabrook method for evolution equations with a small parameter: a technique of approximating pseudopotentials

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Abstract. An approach is suggested to construct approximate solutions for nonlinear partial differential equations non-integrable by the inverse scattering transform. The method is applied to find an approximate Lax pair and conservation laws for the Kawahara equation.

Wahlquist and Estabrook [1] introduced the concept of pseudopotentials into the study of nonlinear partial differential equations. The efficiency of this method in deriving Bäcklund transformations, conservation laws and Lax pairs is shown in [2–4]. However, its applicability is mainly restricted by the equations integrable by means of the inverse scattering transform (IST). As a rule, these integrable models are constructed assuming weak nonlinearities, and the terms with a small parameter are not taken into account in the describing physical processes. The pseudopotentials of these models can be considered approximate for the original 'non-integrable' models. This fact naturally leads to the question whether an approach similar to the Wahlquist–Estabrook (WE) method could be developed to find analogies of the exact pseudopotentials taking into account terms with small parameters in the initial system.

This work is an attempt to generalize the WE method for non-integrable equations with a small parameter close to the integrable ones in order to construct the analogies of Bäcklund transformtions, conservation laws, etc.

Let us consider a system of equations determining the evolution of the function $\boldsymbol{q}(X, t)$ (vectorial in general) with respect to x and t:

$$\begin{aligned} \boldsymbol{q}_x &= \boldsymbol{P}(x, t, v, \ldots, v_{lx}, \boldsymbol{q}) \\ \boldsymbol{q}_t &= \boldsymbol{Q}(x, t, v, \ldots, v_{lx}, \boldsymbol{q}) \end{aligned} \qquad l \in N \tag{1}$$

where $v(x, t)$ is a new function.

The integrability condition of (1)

$$\boldsymbol{q}_{xt} - \boldsymbol{q}_{tx} = 0 \tag{2}$$

clearly imposes a restriction on the type of v.

The requirement (2) is in fact a set of k differential equations relating the function v to the components of the vector $\boldsymbol{q}$; k is its dimension. At $k > 1$, the equations can be incompatible for some $\boldsymbol{P}$ and $\boldsymbol{Q}$. On the other hand, for some special $\boldsymbol{P}$ and $\boldsymbol{Q}$ (2) can be presented as

$$L(v_t - K(v, \ldots, v_{nx})) = 0 \qquad n \in N \tag{3}$$

where L is some linear differential operator with vectorial coefficients.

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Definition 1. The function $\boldsymbol{q}$ in (1) is called a pseudopotential (exact) [2] for the evolution differential equation

$$v_t = K(v, \ldots, v_{nx}) \qquad n \in N$$

if (1) is integrable, and the condition (2) can be presented in the form (3).

Let us take an evolution differential equation with a small parameter ε

$$u_t = K(u, u_x, \ldots, u_{nx}, \varepsilon) \qquad n \in N \qquad |\varepsilon| \ll 1. \tag{4}$$

Definition 2. The function $\boldsymbol{q}$ in (1) is called an approximating pseudopotential (AP) of the mth order for equation (4) if it is the solution of (1), the integrability condition (2) is compatible and is of the form

$$L(v_t - K(v, \ldots, v_{nx}, \varepsilon)) + \mathrm{O}(\varepsilon^{m+1}) = 0 \qquad n, m \in N. \tag{5}$$

It is clear that (4) has an exact pseudopotential if and only if it has an AP of an arbitrary order. On the other hand, for the scalar AP of mth order to exist it is sufficient that (2) is of the form (5).

Below we consider 'the first kind pseudopotentials' [3], i.e. $\boldsymbol{P}$ and $\boldsymbol{Q}$ are taken such that

$$\boldsymbol{P} = \boldsymbol{P}(v, \boldsymbol{q}) \qquad \boldsymbol{Q} = \boldsymbol{Q}(v, \ldots, v_{lx}, \boldsymbol{q}) \qquad l \in N$$

and the operator L is of the form

$$L = \frac{\partial \boldsymbol{P}}{\partial v}.$$

Let $\boldsymbol{P}$ and $\boldsymbol{Q}$ read as

$$\boldsymbol{P} = \boldsymbol{P}_0 + \varepsilon \boldsymbol{P}_1 + \ldots \qquad \boldsymbol{Q} = \boldsymbol{Q}_0 + \varepsilon \boldsymbol{Q}_1 + \ldots .$$

Assume that the function v satisfies (4) with the accuracy of $\mathrm{O}(\varepsilon^{m+1})$, and make the integrability condition (2) of the form

$$\frac{\partial}{\partial t} \boldsymbol{P} - \frac{\partial}{\partial x} \boldsymbol{Q} + [\boldsymbol{P}, \boldsymbol{Q}] = 0$$

$$[\boldsymbol{P}, \boldsymbol{Q}] = \boldsymbol{Q} \frac{\partial}{\partial \boldsymbol{q}} \boldsymbol{P} - \boldsymbol{P} \frac{\partial}{\partial \boldsymbol{q}} \boldsymbol{Q}$$

be met with the same accuracy. Then equating the coefficients of each order in ε up to ε^m to zero, we have

$$\begin{aligned} &\sum_{j=0}^{i} \frac{\partial \boldsymbol{P}_j}{\partial v} F_{i-j} - \frac{\partial}{\partial x} \boldsymbol{Q}_i + \sum_{j=0}^{i} [\boldsymbol{P}_j, \boldsymbol{Q}_{i-j}] = 0 \qquad i = \overline{0, m} \\ &F_j = \frac{1}{j!} \frac{\partial^j K}{\partial \varepsilon^j}\bigg|_{\varepsilon=0} \qquad \forall j. \end{aligned} \tag{6}$$

Proposition 1. Let

$$\boldsymbol{P} = \sum_{i=0}^{m} \varepsilon^i \boldsymbol{P}_i + \boldsymbol{R}_1 \qquad \boldsymbol{Q} = \sum_{i=0}^{m} \varepsilon^i \boldsymbol{Q}_i + \boldsymbol{R}_2$$

where

$$\boldsymbol{R}_1 = \mathrm{O}(\varepsilon^{m+1}) \qquad \boldsymbol{R}_2 = \mathrm{O}(\varepsilon^{m+1})$$

and $\boldsymbol{P}$ and $\boldsymbol{Q}$ are chosen to satisfy (6), then (1) determines the AP of the mth order for (4) if $\boldsymbol{R}_1$, $\boldsymbol{R}_2$ are taken such that the integrability condition (2) is compatible.

Proof. It directly follows from definition 2. □

Corollary 1. When $\dim q = 1$ (i.e. for a scalar pseudopotential), R_1 and R_2 can be arbitrary functions of the $O(\varepsilon^{m+1})$ order.

Proposition 2. (*a*) The equations (6) for P_i and Q_i ($i \neq 0$) are linear and recursively depend on P_j and Q_j ($j < i$)
(*b*) The equation for P_0 and Q_0 is of the type

$$\frac{\partial P_0}{\partial v} K|_{\varepsilon=0} - \frac{\partial}{\partial x} Q_0 + [P_0, Q_0] = 0.$$

Proof. These facts directly follow from (6). □

Corollary 2. For an AP of a certain order for (4) to exist, it is necessary that there exists an exact pseudopotential for the equation

$$u_t = K|_{\varepsilon=0}.$$

The procedure of finding P_i and Q_i is fully analogous to that described in [2] and reduces to the problem of the Lie algebra theory, namely, to the problem on the existence of some Lie algebra (Abelian or non-Abelian) compatible with the commutation relations obtained from (6).

In the same way the APs analogous to the exact pseudopotentials that give rise to conservation laws, Bäcklund transformations and the IST can be obtained. Adiabatic invariants serve here as analogues of conserved densities.

As an example let us construct the APs of the Kawahara equation resulting in both an approximate Lax pair and adiabatic invariants. For the sake of simplicity only one-dimensional pseudopotentials will be considered. The system (1) will also be called 'a pseudopotential' for short.

The Kawahara equation is well known in the theory of continuous medium. It arises in various applications, e.g. magneto-acoustic waves, waves in shallow water, and waves in nonlinear electric circuits (a review may be found in [5]) and reads as

$$u_t + 6uu_x + u_{xxx} + \varepsilon u_{xxxxx} = 0. \tag{7}$$

Equation (7) can be shown to have no non-Abelian pseudopotentials (at least, no scalar ones). However, in what follows the Kawahara equation is proved to have the non-Abelian APs of the first order on the assumption $|\varepsilon| \ll 1$. This assumption corresponds to the case of a long-wave perturbation described by (7).

In case of $\varepsilon = 0$ (7) is transformed into the Korteweg-de Vries equation (KdV) integrable by means of the IST. The latter can be obtained from the following exact pseudopotential for the KdV equation [4]

$$\begin{aligned} q_x &= -q^2 - u + \lambda \\ q_t &= 2q^2 u + 4\lambda q^2 - 2qu_x + 2u^2 + 2\lambda u + u_{xx} - 4\lambda^2 u \end{aligned} \tag{8}$$

where λ is the spectral parameter.

Let us construct an AP *of the equation* (7) *corresponding to* (8). For this purpose it is necessary to find an AP of the type

$$\begin{aligned} q_x &= -q^2 - u + \lambda + \varepsilon P_1(u, q) + O(\varepsilon^2) \\ q_t &= 2q^2 u + 4\lambda q^2 - 2qu_x + 2u^2 + 2\lambda u + u_{xx} - 4\lambda^2 u \\ &\quad + \varepsilon Q_1(u, u_x, u_{xx}, u_{xxx}, u_{xxxx}, q) + O(\varepsilon^2). \end{aligned} \tag{9}$$

Applying the technique described in [2] to (6), one finds P_1, Q_1 and (9) are of the form of the Riccati equation

$$P_1=\tfrac{1}{3}(-5q^2u+4q^2\lambda-5u^2+11\lambda u-4\lambda^2)$$
$$Q_1=\tfrac{1}{3}(18q^2u^2-4q^2u\lambda+11q^2u_{xx}+4quu_x-4\lambda qu_x-6qu_{xxx}+18u^3 -26u^3\lambda+4uu_{xx}-4\lambda^2u_x-2(u_x)^2-5\lambda u_{xx}-3u_{xxxx}). \qquad (10)$$

Presenting q as

$$q=\frac{\partial}{\partial x}\ln\psi(x,t)$$

(ψ is the spectral function) and following [4], one gets a pair of linear equations for ψ from (9) and (10), which is analogous to the Lax pairs for integrable equations

$$\psi_{xx}-\varepsilon\tfrac{5}{3}u_x\psi_x+(\tfrac{1}{3}\varepsilon(10u^2-20u\lambda+3u+8\lambda^2)+u-\lambda)\psi=0$$
$$\psi_t+(\tfrac{1}{3}\varepsilon(11u_{xx}+8u^2-16u\lambda+16\lambda^2)+2u+4\lambda)\psi_x +(\varepsilon(\tfrac{2}{3}uu_x-\tfrac{2}{3}\lambda u_x-u_{xxx})-u_x)\psi=0. \qquad (11)$$

(The terms of the $O(\varepsilon^2)$ order are missing.)

Clearly, at $\varepsilon=0$ these equations is transformed into the well known Lax pair of the KdV equation [6]. Note that the set (9) is a generalized Miura transformation well known for the KdV [6].

Equations (11) can be reduced to the presentation suggested by Ablowitz *et al* [6]

$$\mathbf{q}_x=X\mathbf{q}+O(\varepsilon^2) \qquad \mathbf{q}_t=T\mathbf{q}+O(\varepsilon^2) \qquad (12)$$

where X and T are some matrices of the final dimension, and

$$X=\sum_{i=0}^{m}\lambda^iX_i \qquad T=\sum_{i=0}^{n}\lambda^iT_i \qquad m,n\in N.$$

(X_i and T_i are new matrices of the same dimension which are independent of the spectral parameter λ.)

The Lax pair (11) and the presentation (12) can be used to solve approximately (7) by means of the IST and to settle the problem on the deformation of N-soliton solutions of the KdV equation on account of the small contribution of the fifth derivative in (7) by the technique similar to that suggested in [7]. A detailed treatment of the problems as well as the consideration of any AP of higher orders is planned to be conducted by the authors in the future.

Let us discuss in more detail finding the analogies of conservation laws and, respectively, adiabatic invariants. In the general case, one-dimensional Abelian APs can be related to approximate conservation laws. For equation (7) they can be derived recurrently. For this purpose it is necessary to construct an analgoue of the exact pseudopotential for the KdV equation

$$q_x=\lambda q-q^2-u \qquad q_t=(-\lambda^2q+\lambda u-2qu+u_x)_x$$

that leads to the KdV hierarchy of conservation laws [6]. The AP for (7) is

$$q_x=(\lambda q-q^2-u)+\varepsilon(-\tfrac{1}{4}\lambda^4+\tfrac{5}{3}\lambda^2q^2-\tfrac{10}{3}\lambda q^3-\tfrac{5}{3}q^4+\tfrac{10}{3}q^2u-\tfrac{5}{3}u^2)+O(\varepsilon^2) \qquad (13)$$

$$q_t=\frac{\partial}{\partial x}(-\lambda^2q+\lambda u-2qu+u_x)+\tfrac{1}{6}\varepsilon\frac{\partial}{\partial x}(6\lambda^3u+8\lambda^2qu+6\lambda^2u_x-2\lambda u^2 +6\lambda u_{xx}-20q^2u_x-16qu^2-12qu_{xx}-4uu_x+6u_{xxx})+O(\varepsilon^2). \qquad (14)$$

Presenting q as

$$q(x, t) = q_0(x, t) + \varepsilon q_1(x, t) + O(\varepsilon^2) \tag{15}$$

and the functions q_0, q_1 as

$$q_0(x, t) = \sum_{i=1}^{\infty} \lambda^i W_{0i}(x, t) \qquad q_1(x, t) = \sum_{j=-3}^{\infty} \lambda^j W_{1j}(x, t).$$

($W_{0i}(x, t)$ and $W_{1j}(x, t)$ are new functions) then substituting (15) into (13), (14), omitting the terms of $O(\varepsilon^2)$ order, and making the coefficients at the powers of λ equal to zero, from (13) one gets the recurrence formulas to determine W_{0i} and W_{1i} with respect to W_{0j} and W_{1j} ($j < i$). (The corresponding adiabatic invariants are of the form

$$\rho_j = \int_{-\infty}^{+\infty} W_{0j}\, dx + \varepsilon \int_{-\infty}^{+\infty} W_{1j}\, dx \qquad j = \overline{1, \infty}.$$

It is obvious that the first terms determine conserved densities of the KdV equation.) Accordingly equation (14) determines approximate conservation laws. The first four non-trivial adiabatic invariants are as follows:

$$\rho_1 = \int_{-\infty}^{+\infty} u\, dx \tag{16}$$

$$\rho_3 = \int_{-\infty}^{+\infty} u^2\, dx \tag{17}$$

$$\rho_5 = \int_{-\infty}^{+\infty} (-u_x^2 + 2u^3 + \varepsilon u_{xx}^2)\, dx \tag{18}$$

$$\rho_7 = \int_{-\infty}^{+\infty} ((-22u_x^2 u - 6u_{xx}u^2 + 5u^4 + u_{xx}^2) + \varepsilon(-\tfrac{40}{21}u_{xxx}^2 + 10u_{xx}^2 u + \tfrac{9908}{21}u_{xx}u_x^2))\, dx.$$

Note that ρ_1, ρ_3 and ρ_5 are identical to the exact conserved densities of (7). Moreover, the Kawahara equation represents the Hamiltonian system

$$\frac{\partial}{\partial t} u = \frac{\partial}{\partial x}\left(\frac{\delta H}{\delta u}\right) \qquad H = -\tfrac{1}{2}\rho_5 .$$

The conserved densities (16)–(18) may be used to establish global existence of solutions (7) for the Cauchy problem in Sobolev spaces [8, 9], in particular, to prove stability of the solitary wave type solution obtained in [10, 11] for equation (7).

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